
SUPPLEMENTARY MATERIAL TO

How to Hedge an Option Against an Adversary: Black-Scholes Pricing is Minimax Optimal

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A Proof of Lemma 3

For each $1 \leq i \leq n$, the random variable $\log(1 + R_{i,n})$ has mean and variance, respectively,

$$\mu_n = \frac{1}{2} \log \left(1 - \frac{c}{n} \right) \quad \text{and} \quad \sigma_n^2 = \frac{1}{4} \log^2 \left(\frac{\sqrt{n} + \sqrt{c}}{\sqrt{n} - \sqrt{c}} \right).$$

We now define

$$X_{i,n} := \frac{\log(1 + R_{i,n}) - \mu_n}{\sigma_n \sqrt{n}}, \tag{13}$$

so $X_{1,n}, \dots, X_{n,n}$ are i.i.d. random variables with $\mathbb{E}[X_{i,n}] = 0$ and $\sum_{i=1}^n \mathbb{E}[X_{i,n}^2] = 1$. Recalling that $R_{i,n} \in \{\pm\sqrt{c/n}\}$, we see that the two possible values for $X_{i,n}$ both approach 0 as $n \rightarrow \infty$. This means for any $\epsilon > 0$ we can find a sufficiently large n such that $|X_{i,n}| < \epsilon$ for all $1 \leq i \leq n$. In particular, this implies the Lindeberg condition for the triangular array $(X_{i,n}, 1 \leq i \leq n)$: for all $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \mathbb{E}[X_{i,n}^2 \mathbf{1}\{|X_{i,n}| > \epsilon\}] = 0.$$

Thus, by the Lindeberg central limit theorem [4, Theorem 3.4.5], we have the convergence in distribution $\sum_{i=1}^n X_{i,n} \xrightarrow{d} Z$, where $Z \sim \mathcal{N}(0, 1)$ is a standard Gaussian random variable.

Clearly $\mu_n \rightarrow 0$ as $n \rightarrow \infty$. Furthermore, one can easily verify that by the L'Hôpital's rule,

$$\lim_{n \rightarrow \infty} \sigma_n \sqrt{n} = \sqrt{c} \quad \text{and} \quad \lim_{n \rightarrow \infty} n\mu_n = -\frac{c}{2}.$$

Therefore, from the convergence $\sum_{i=1}^n X_{i,n} \xrightarrow{d} Z$ and recalling the definition (13) of $X_{i,n}$, we also obtain

$$\sum_{i=1}^n \log(1 + R_{i,n}) = \sum_{i=1}^n (\mu_n + \sigma_n \sqrt{n} X_{i,n}) = n\mu_n + \sigma_n \sqrt{n} \sum_{i=1}^n X_{i,n} \xrightarrow{d} \frac{-c}{2} + \sqrt{c}Z.$$

In particular, by the continuous mapping theorem,

$$\prod_{i=1}^n (1 + R_{i,n}) \xrightarrow{d} \exp \left(-\frac{c}{2} + \sqrt{c}Z \right) \stackrel{d}{=} G(c).$$

We now want to show that we also have convergence in expectation when g is an L -Lipschitz function, namely, that $\mathbb{E}[g(S \cdot \prod_{i=1}^n (1 + R_{i,n}))] \rightarrow \mathbb{E}[g(S \cdot G(c))]$. Without loss of generality (by replacing $g(x)$ by $\hat{g}(x) = g(S \cdot x) - g(0)$) we may assume $S = 1$ and $g(0) = 0$. For simplicity, let $S_n = \prod_{i=1}^n (1 + R_{i,n})$. For each $M > 0$ define the continuous bounded function $g_M(x) = \min\{g(x), M\}$. The convergence in distribution $S_n \xrightarrow{d} G(c)$ gives us

$$\lim_{n \rightarrow \infty} \mathbb{E}[g_M(S_n)] = \mathbb{E}[g_M(G(c))] \quad \text{for all } M > 0. \quad (14)$$

Since $g_M \uparrow g$ pointwise, by the monotone convergence theorem we also have

$$\lim_{M \rightarrow \infty} \mathbb{E}[g_M(G(c))] = \mathbb{E}[g(G(c))]. \quad (15)$$

Now observe that $\mathbb{E}[S_n] = 1$ and $\mathbb{E}[S_n^2] = (1 + c/n)^n \leq \exp(c)$. Since $g(0) = 0$ and g is L -Lipschitz, we have $g(x) \leq Lx$ for all $x \geq 0$. In particular, $\mathbb{E}[g(S_n)^2] \leq L^2 \mathbb{E}[S_n^2] \leq L^2 \exp(c)$. Moreover, by Markov's inequality,

$$\mathbb{P}(g(S_n) > M) \leq \mathbb{P}\left(S_n > \frac{M}{L}\right) \leq \frac{\mathbb{E}[S_n]}{M/L} = \frac{L}{M}.$$

Therefore, for each n and for all $M > 0$, by Cauchy-Schwarz inequality,

$$\begin{aligned} |\mathbb{E}[g(S_n)] - \mathbb{E}[g_M(S_n)]| &= \mathbb{E}[(g(S_n) - M) \cdot \mathbf{1}\{g(S_n) > M\}] \\ &\leq \mathbb{E}[g(S_n) \cdot \mathbf{1}\{g(S_n) > M\}] \\ &\leq \mathbb{E}[g(S_n)^2]^{1/2} \mathbb{P}(g(S_n) > M)^{1/2} \\ &\leq (L^3 \exp(c)/M)^{1/2}. \end{aligned}$$

Since the final bound does not involve n , this shows that $\lim_{M \rightarrow \infty} \mathbb{E}[g_M(S_n)] \rightarrow \mathbb{E}[g(S_n)]$ uniformly in n . This allows us to interchange the order of the limit operations below, which, together with (14) and (15), give us our desired result:

$$\lim_{n \rightarrow \infty} \mathbb{E}[g(S_n)] = \lim_{n \rightarrow \infty} \lim_{M \rightarrow \infty} \mathbb{E}[g_M(S_n)] = \lim_{M \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbb{E}[g_M(S_n)] = \lim_{M \rightarrow \infty} \mathbb{E}[g_M(G(c))] = \mathbb{E}[g(G(c))].$$

This completes the proof of Lemma 3.

B Proofs of Lemma 5 and Lemma 7

B.1 Proof of Lemma 5

Lemma 5 essentially follows from the definition of $\alpha^{(n)}$.

Proof of Lemma 5. We proceed by induction on m . For the base case $m = 0$, we use Jensen's inequality and the fact that $\mathbb{E}[G(c)] = 1$:

$$V_\zeta^{(n)}(S; c, 0) = g(S) = g(S \cdot \mathbb{E}[G(c)]) \leq \mathbb{E}[g(S \cdot G(c))] = U(S, c).$$

Now assume the statement (10) holds for $m - 1$. Then for m ,

$$\begin{aligned} V_\zeta^{(n)}(S; c, m) &= \inf_{\Delta \in \mathbb{R}} \sup_{|r| \leq \min\{\zeta, \sqrt{c}\}} -\Delta r + V_\zeta^{(n)}(S + Sr; c - r^2, m - 1) \\ &\leq \inf_{\Delta \in \mathbb{R}} \sup_{|r| \leq \min\{\zeta, \sqrt{c}\}} -\Delta r + U(S + Sr, c - r^2) + \alpha^{(n)}(S + Sr, c - r^2, m - 1) \\ &\leq \sup_{|r| \leq \min\{\zeta, \sqrt{c}\}} -r S U_S(S, c) + U(S + Sr, c - r^2) + \alpha^{(n)}(S + Sr, c - r^2, m - 1) \\ &= \sup_{|r| \leq \min\{\zeta, \sqrt{c}\}} U(S, c) + \epsilon_r(S, c) + \alpha^{(n)}(S + Sr, c - r^2, m - 1) \\ &= U(S, c) + \alpha^{(n)}(S, c, m). \end{aligned}$$

The first line is from the definition (5); the second line is using the inductive hypothesis that (10) holds for $m - 1$; the third line is from substituting the choice $\Delta = S U_S(S, c)$; the fourth line is from the definition of ϵ_r ; and the last line is from the definition of $\alpha^{(n)}(S, c, m)$. \square

B.2 Proof of Lemma 7

For completeness, we provide a more detailed proof of Lemma 7.

Proof of Lemma 7. Unrolling the inductive definition (9), we can write

$$\alpha^{(n)}(S, c) = \sup_{\substack{r_1, \dots, r_n \\ |r_m| \leq \zeta, \sum_{m=1}^n r_m^2 \leq c}} f(r_1, \dots, r_n),$$

where f is the function

$$f(r_1, \dots, r_n) = \sum_{m=1}^n \epsilon_{r_m} \left(S \prod_{i=1}^{m-1} (1 + r_i), c - \sum_{i=1}^{m-1} r_i^2 \right).$$

Let (r_1, \dots, r_n) be such that $|r_m| \leq \zeta$ and $\sum_{m=1}^n r_m^2 \leq c$. We will show that $f(r_1, \dots, r_n) \leq (18c + 8/\sqrt{2\pi}) LK \zeta^{1/4}$.

Assume for now that $\zeta \leq c^2$. Let $0 \leq n_* \leq n$ be the largest index such that

$$\sum_{m=1}^{n_*} r_m^2 \leq c - \sqrt{\zeta}.$$

We split the analysis into two parts.

For $1 \leq m \leq \min\{n, n_* + 1\}$: We want to apply the bound in Lemma 6, so let us verify that the conditions in Lemma 6 are satisfied. Clearly $|r_m| \leq \zeta \leq 1/16$. Moreover, since $c - \sum_{i=1}^{m-1} r_i^2 \geq c - \sum_{i=1}^{n_*} r_i^2 \geq \sqrt{\zeta}$ and $\zeta \leq 1/16$, we also have

$$|r_m| \leq \zeta \leq \frac{\zeta^{1/4}}{8} \leq \frac{\sqrt{c - \sum_{i=1}^{m-1} r_i^2}}{8}.$$

Therefore, by (11) from Lemma 6,

$$\begin{aligned} \epsilon_{r_m} \left(S \prod_{i=1}^{m-1} (1 + r_i), c - \sum_{i=1}^{m-1} r_i^2 \right) &\leq 16LK \left(\max \left\{ \left(c - \sum_{i=1}^{m-1} r_i^2 \right)^{-3/2}, \left(c - \sum_{i=1}^{m-1} r_i^2 \right)^{-1/2} \right\} |r_m|^3 \right. \\ &\quad \left. + \max \left\{ \left(c - \sum_{i=1}^{m-1} r_i^2 \right)^{-2}, \left(c - \sum_{i=1}^{m-1} r_i^2 \right)^{-1/2} \right\} r_m^4 \right) \\ &\leq 16LK \left(\max\{\zeta^{-3/4}, \zeta^{-1/4}\} |r_m|^3 + \max\{\zeta^{-1}, \zeta^{-1/4}\} r_m^4 \right) \\ &= 16LK \left(\zeta^{-3/4} |r_m|^3 + \zeta^{-1} r_m^4 \right) \quad (\text{since } \zeta < 1) \\ &\leq 16LK \left(\zeta^{1/4} r_m^2 + \zeta r_m^2 \right) \quad (\text{since } |r_m| \leq \zeta) \\ &\leq 16LK \left(\zeta^{1/4} r_m^2 + \zeta^{1/4} \frac{1}{16^{3/4}} r_m^2 \right) \quad (\text{since } \zeta \leq 1/16) \\ &= 18LK \zeta^{1/4} r_m^2. \end{aligned}$$

Summing over $1 \leq m \leq \min\{n, n_* + 1\}$ gives us

$$\sum_{m=1}^{\min\{n, n_*+1\}} \epsilon_{r_m} \left(S \prod_{i=1}^{m-1} (1 + r_i), c - \sum_{i=1}^{m-1} r_i^2 \right) \leq 18LK \zeta^{1/4} \sum_{m=1}^{\min\{n, n_*+1\}} r_m^2 \leq 18LK \zeta^{1/4} c. \quad (16)$$

For $n_* + 2 \leq m \leq n$, if $n_* \leq n - 2$: Without loss of generality we may assume $r_n \neq 0$, for if $r_n = 0$, then the term depending on r_n does not affect $f(r_1, \dots, r_n)$ since

$$\epsilon_{r_n} \left(S \prod_{i=1}^{n-1} (1 + r_i), c - \sum_{i=1}^{n-1} r_i^2 \right) = 0,$$

so we can remove r_n and only consider $n_* + 2 \leq m \leq n - 1$. From the definition of n_* we see that $\sum_{m=1}^{n_*+1} r_m^2 > c - \sqrt{\zeta}$, and since $\sum_{m=1}^n r_m^2 \leq c$, this implies

$$\sum_{m=n_*+2}^n r_m^2 \leq c - \sum_{m=1}^{n_*+1} r_m^2 < c - (c - \sqrt{\zeta}) = \sqrt{\zeta}. \quad (17)$$

Note also that for each $n_* + 2 \leq m \leq n$,

$$0 < r_n^2 \leq \sum_{i=m}^n r_i^2 \leq c - \sum_{i=1}^{m-1} r_i^2 \leq c - \sum_{i=1}^{n_*+1} r_i^2 \leq \sqrt{\zeta} \leq \frac{1}{4},$$

so by (12) from Lemma 6,

$$\epsilon_{r_m} \left(S \prod_{i=1}^{m-1} (1 + r_i), c - \sum_{i=1}^{m-1} r_i^2 \right) \leq \frac{4LK}{\sqrt{2\pi}} \cdot \frac{r_m^2}{\sqrt{c - \sum_{i=1}^{m-1} r_i^2}} \leq \frac{4LK}{\sqrt{2\pi}} \cdot \frac{r_m^2}{\sqrt{\sum_{i=m}^n r_i^2}}.$$

Therefore, by applying Lemma 8 below to $x_i = r_{n_*+1+i}^2$, we see that

$$\begin{aligned} \sum_{m=n_*+2}^n \epsilon_{r_m} \left(S \prod_{i=1}^{m-1} (1 + r_i), c - \sum_{i=1}^{m-1} r_i^2 \right) &\leq \frac{4LK}{\sqrt{2\pi}} \sum_{m=n_*+2}^n \frac{r_m^2}{\sqrt{\sum_{i=m}^n r_i^2}} \\ &\leq \frac{8LK}{\sqrt{2\pi}} \left(\sum_{m=n_*+2}^n r_m^2 \right)^{1/2} \leq \frac{8LK}{\sqrt{2\pi}} \zeta^{1/4}, \end{aligned} \quad (18)$$

where the last inequality follows from (17). Combining (16) and (18) gives us the desired conclusion.

Now if $\zeta > c^2$, then the argument in the second case above (for $n_* + 2 \leq m \leq n$) still holds with n_* set to be -1 , so we still get the same conclusion. \square

It now remains to prove the following result, which we use at the end of the proof of Lemma 7.

Lemma 8. For $x_1, \dots, x_k \geq 0$ with $x_k > 0$, we have

$$\sum_{i=1}^k \frac{x_i}{\sqrt{x_i + x_{i+1} + \dots + x_k}} \leq 2 \left(\sum_{i=1}^k x_i \right)^{1/2}.$$

Proof. Let \mathcal{L}_k denote the objective function that we wish to bound,

$$\mathcal{L}_k(x_1, \dots, x_k) = \sum_{i=1}^k \frac{x_i}{\sqrt{x_i + x_{i+1} + \dots + x_k}},$$

and note that for any $t > 0$,

$$\mathcal{L}_k(tx_1, \dots, tx_k) = \sqrt{t} \mathcal{L}_k(x_1, \dots, x_k), \quad (19)$$

For each $k \in \mathbb{N}$, let Δ_k denote the unit simplex in \mathbb{R}^k with $x_k > 0$,

$$\Delta_k = \left\{ (x_1, \dots, x_k) : x_1, \dots, x_{k-1} \geq 0, x_k > 0, \sum_{i=1}^k x_i = 1 \right\},$$

and let η_k denote the supremum of the function \mathcal{L}_k over $x \in \Delta_k$. Given $x = (x_1, \dots, x_k) \in \Delta_k$, define $y = (y_1, \dots, y_{k-1})$ by $y_i = x_{i+1}/(1 - x_1)$, so $y \in \Delta_{k-1}$. Then we can write

$$\begin{aligned} \mathcal{L}_k(x_1, \dots, x_k) &= \frac{x_1}{\sqrt{x_1 + \dots + x_k}} + \mathcal{L}_{k-1}(x_2, \dots, x_k) \\ &= x_1 + \sqrt{1 - x_1} \mathcal{L}_{k-1}(y_1, \dots, y_{k-1}) \\ &\leq x_1 + \sqrt{1 - x_1} \eta_{k-1}, \end{aligned}$$

where the second equality is from (19) and the last inequality is from the definition of η_{k-1} . The function $x_1 \mapsto x_1 + \sqrt{1 - x_1} \eta_{k-1}$ is concave and maximized at $x_1^* = 1 - \eta_{k-1}^2/4$, giving us

$$\mathcal{L}_k(x_1, \dots, x_k) \leq x_1^* + \sqrt{1 - x_1^*} \eta_{k-1} = 1 - \frac{\eta_{k-1}^2}{4} + \sqrt{\frac{\eta_{k-1}^2}{4}} \eta_{k-1} = 1 + \frac{\eta_{k-1}^2}{4}.$$

Taking the supremum over $x \in \Delta_k$ gives us the recursion

$$\eta_k \leq 1 + \frac{\eta_{k-1}^2}{4},$$

which, along with the base case $\eta_1 = 1$, easily implies $\eta_k \leq 2$ for all $k \in \mathbb{N}$. Now given $x_1, \dots, x_k \geq 0$ with $x_k > 0$, let $x' = (tx_1, \dots, tx_k)$ with $t = 1/(x_1 + \dots + x_k)$, so $x' \in \Delta_k$. Then using (19) and the bound $\eta_k \leq 2$, we get

$$\mathcal{L}_k(x_1, \dots, x_k) = \frac{1}{\sqrt{t}} \mathcal{L}_k(tx_1, \dots, tx_k) \leq \eta_k \left(\sum_{i=1}^k x_i \right)^{1/2} \leq 2 \left(\sum_{i=1}^k x_i \right)^{1/2},$$

as desired. \square

C Proof of Lemma 6

In this section we provide a proof of Lemma 6. Throughout the rest of this paper, we use the following notation for the higher-order partial derivatives of U ,

$$U_{S^a c^b}(S, c) = \frac{\partial^{a+b} U(S, c)}{\partial S^a \partial c^b}, \quad a, b \in \mathbb{N}_0.$$

We will use the following bounds on U_{S^2} , U_{S^3} , and U_{S^4} , which we prove in Appendix D. These bounds are where we use the crucial assumptions that the payoff function g is convex, L -Lipschitz, and K -linear.

Lemma 9. *Let $g: \mathbb{R}_0 \rightarrow \mathbb{R}_0$ be a convex, L -Lipschitz, K -linear function. Then for all $S, c > 0$,*

$$|U_{S^2}(S, c)| \leq \frac{2LK}{\sqrt{2\pi}} \cdot \frac{1}{S^2 \sqrt{c}} \quad (20)$$

$$|U_{S^3}(S, c)| \leq 7LK \cdot \frac{\max\{c^{-3/2}, c^{-1/2}\}}{S^3}, \quad (21)$$

$$|U_{S^4}(S, c)| \leq 28LK \cdot \frac{\max\{c^{-2}, c^{-1/2}\}}{S^4}. \quad (22)$$

We will also use the following property of the function U .

Lemma 10. *The function $U(S, c)$ is convex in S and non-decreasing in c .*

Proof. For each fixed $c \geq 0$ and for each realization of the random variable $G(c) > 0$, the function $S \mapsto g(S \cdot G(c))$ is convex. Therefore, $U(S, c)$ is convex in S , being a nonnegative linear combination of convex functions. In particular, this implies $U_{S^2}(S, c) \geq 0$. So by the Black-Scholes equation (6), we also have $U_c(S, c) = \frac{1}{2} S^2 U_{S^2}(S, c) \geq 0$. \square

We are now ready to prove Lemma 6. For clarity, we divide the proof into two parts: we first prove the bound (12), then prove the bound (11).

Proof of (12) in Lemma 6. Recall that $U(S, c)$ is non-decreasing in c by Lemma 10. Then by the Taylor remainder theorem, we can write

$$\begin{aligned} \epsilon_r(S, c) &= U(S + Sr, c - r^2) - U(S, c) - rSU_S(S, c) \\ &\leq U(S + Sr, c) - U(S, c) - rSU_S(S, c) \\ &= \frac{1}{2} r^2 S^2 U_{S^2}(S + S\xi, c) \end{aligned}$$

where ξ is some value between 0 and r . Since $|\xi| \leq |r| \leq \sqrt{c} \leq 1/2$, we have $(1 + \xi)^2 \geq 1/4$. Moreover, from (20) in Lemma 9, we have

$$|(1 + \xi)^2 S^2 U_{S^2}(S + S\xi, c)| \leq \frac{2LK}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{c}}.$$

Combining the bounds above gives us

$$\epsilon_r(S, c) \leq \frac{1}{2} \frac{r^2}{(1 + \xi)^2} |(1 + \xi)^2 S^2 U_{S^2}(S + S\xi, c)| \leq \frac{4LK}{\sqrt{2\pi}} \cdot \frac{r^2}{\sqrt{c}},$$

as desired. \square

Proof of (11) in Lemma 6. Fix $S, c > 0$, and consider the function

$$f(r) = U(S + Sr, c - r^2), \quad |r| \leq \sqrt{c}.$$

By repeatedly applying the Black-Scholes differential equation (6), we can easily verify that $f(0) = U(S, c)$, $f'(0) = SU_S(S, c)$, and

$$\begin{aligned} f''(r) = & p_2(r) r S^2 U_{S^2}(S + Sr, c - r^2) + p_3(r) (1 + r)^2 r S^3 U_{S^3}(S + Sr, c - r^2) \\ & + (1 + r)^4 r^2 S^4 U_{S^4}(S + Sr, c - r^2), \end{aligned} \quad (23)$$

where p_2, p_3 are the polynomials $p_2(r) = 2r^3 + 4r^2 - 3r - 6$ and $p_3(r) = 4r^2 + 4r - 2$.

Noting that we can write

$$\epsilon_r(S, c) = f(r) - f(0) - f'(0)r,$$

another application of Taylor's remainder theorem allows us to write

$$\epsilon_r(S, c) = \frac{1}{2} f''(\xi) r^2$$

for some ξ lying between 0 and r . It is easy to verify that we have

$$\left| \frac{p_2(\xi)}{(1 + \xi)^2} \right| \leq 7, \quad \left| \frac{p_3(\xi)}{(1 + \xi)} \right| \leq 3 \quad \text{for all } |\xi| \leq |r| \leq \frac{1}{16}.$$

Moreover, since $\xi^2 \leq r^2 \leq c/64$, we have $c - \xi^2 \geq \frac{63}{64}c$. Then from the bound (20) in Lemma 9, we have

$$|(1 + \xi)^2 S^2 U_{S^2}(S + S\xi, c - \xi^2)| \leq \frac{2LK}{\sqrt{2\pi}} \cdot \frac{1}{(c - \xi^2)^{1/2}} \leq \frac{2LK}{\sqrt{2\pi}} \cdot \frac{1}{(\frac{63}{64}c)^{1/2}} \leq LK c^{-1/2}.$$

We also get from the bound (21) in Lemma 9,

$$\begin{aligned} |(1 + \xi)^3 S^3 U_{S^3}(S + S\xi, c - \xi^2)| & \leq 7LK \max\{(c - \xi^2)^{-3/2}, (c - \xi^2)^{-1/2}\} \\ & \leq 7LK \max\left\{\left(\frac{63}{64}c\right)^{-3/2}, \left(\frac{63}{64}c\right)^{-1/2}\right\} \\ & \leq 7LK \left(\frac{64}{63}\right)^{3/2} \max\{c^{-3/2}, c^{-1/2}\} \\ & \leq 8LK \max\{c^{-3/2}, c^{-1/2}\}. \end{aligned}$$

Similarly, the bound (22) in Lemma 9 gives us

$$|(1 + \xi)^4 S^4 U_{S^4}(S + S\xi, c - \xi^2)| \leq 29LK \max\{c^{-2}, c^{-1/2}\}.$$

Applying the bounds above to (23) gives us

$$\begin{aligned} |f''(\xi)| & \leq \left| \frac{p_2(\xi)}{(1 + \xi)^2} \right| \cdot |\xi| \cdot |(1 + \xi)^2 S^2 U_{S^2}(S + S\xi, c - \xi^2)| \\ & \quad + \left| \frac{p_3(\xi)}{(1 + \xi)} \right| \cdot |\xi| \cdot |(1 + \xi)^3 S^3 U_{S^3}(S + S\xi, c - \xi^2)| \\ & \quad + \xi^2 \cdot |(1 + \xi)^4 S^4 U_{S^4}(S + S\xi, c - \xi^2)| \\ & \leq 7LK |r| c^{-1/2} + 24LK |r| \max\{c^{-3/2}, c^{-1/2}\} + 29LK r^2 \max\{c^{-2}, c^{-1/2}\} \\ & \leq 31LK |r| \max\{c^{-3/2}, c^{-1/2}\} + 29LK r^2 \max\{c^{-2}, c^{-1/2}\}. \end{aligned}$$

Therefore, we obtain

$$|\epsilon_r(S, c)| = \frac{1}{2} |f''(\xi)| \cdot r^2 \leq 16LK \left(|r|^3 \max\{c^{-3/2}, c^{-1/2}\} + r^4 \max\{c^{-2}, c^{-1/2}\} \right),$$

as desired. \square

D Proof of the Bounds on the Derivatives (Lemma 9)

In this section we prove the bounds on the higher-order derivatives $U_{S^a}(S, c)$, $a \geq 0$. Proving the bounds in Lemma 9 is more difficult than the analysis that we have done so far, and uses the full force of the assumptions that the payoff function g is convex, L -Lipschitz, and K -linear.

The outline of the proof is as follows. By writing $U(S, c)$ as a convolution, we can write its derivatives $U_{S^a}(S, c)$ as an expectation of $g(S \cdot G(c))$ modulated by certain polynomials (Appendix D.1). The K -linearity of g allows us to approximate g by the European-option payoff function g_{EC} that we encountered in Section 2, so we first prove Lemma 9 for the specific case when the payoff function is g_{EC} (Appendix D.2). We extend the bound on $U_{S^2}(S, c)$ to the general case by dominating the function inside the expectation by another carefully constructed function (Appendix D.3). Finally, we use the approximation of g by g_{EC} to prove the bounds on the higher-order derivatives U_{S^3} and U_{S^4} (Appendix D.4). In particular, Lemma 15 proves the bound (20), and Lemma 18 proves the bounds (21) and (22).

Throughout the rest of this appendix, $Z \sim \mathcal{N}(0, 1)$ denotes a standard Gaussian random variable, and Φ and ϕ denote the cumulative distribution function and the probability density function, respectively, of the standard Gaussian distribution. The symbol $*$ denotes the convolution operator on \mathbb{R} . We also use the fact that $G(c) \stackrel{d}{=} \exp(-\frac{1}{2}c + \sqrt{c}Z)$. Recall that the convexity of g implies differentiability almost everywhere, so we can work with its derivative g' , which is necessarily increasing (since g is convex) and satisfies $|g'(x)| \leq L$ (since g is L -Lipschitz).

Finally, in the proofs below we use the following easy property, which we state without proof.

Lemma 11. *For $f: \mathbb{R} \rightarrow \mathbb{R}$, $Z \sim \mathcal{N}(0, 1)$, and $c \geq 0$, we have*

$$\mathbb{E}[f(Z) \exp(\sqrt{c}Z)] = \exp\left(\frac{c}{2}\right) \mathbb{E}[f(Z + \sqrt{c})],$$

provided all the expectations above exist.

D.1 Formulae for the Derivatives

In this section we show that the partial derivative $U_{S^a}(S, c)$ can be expressed as an expectation of a polynomial modulated by the payoff function g . We define the family of polynomials $p^{[a]}(x, y)$, $a \geq 0$, as follows:

$$\begin{aligned} p^{[0]}(x, y) &= 1 \\ p^{[a+1]}(x, y) &= (x - ay) p^{[a]}(x, y) - p_x^{[a]}(x, y) \quad \text{for } a \geq 1, \end{aligned} \tag{24}$$

where $p_x^{[a]}(x, y) = \partial p^{[a]}(x, y) / \partial x$.

The following is the main result in this section; note that we only assume that g is Lipschitz.

Lemma 12. *Let $g: \mathbb{R}_0 \rightarrow \mathbb{R}_0$ be an L -Lipschitz function. For $a \geq 0$ and $S, c > 0$,*

$$U_{S^a}(S, c) = \frac{1}{S^a c^{a/2}} \mathbb{E} \left[p^{[a]}(Z, \sqrt{c}) \cdot g \left(S \cdot \exp \left(-\frac{c}{2} + \sqrt{c}Z \right) \right) \right],$$

where $Z \sim \mathcal{N}(0, 1)$.

In proving Lemma 12 we will need the following result, which allows us to differentiate the convolution.

Lemma 13. *Fix $c > 0$. Let $g: \mathbb{R}_0 \rightarrow \mathbb{R}_0$ be an L -Lipschitz function, and let $\tilde{g}(x) = g(\exp(x))$. Let $\omega: \mathbb{R} \rightarrow \mathbb{R}$ be given by $\omega(x) = p(x) \phi(x/\sqrt{c})$, where $p(x)$ is a polynomial in x with coefficients involving c . Finally, let $f: \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(r) = (\tilde{g} * \omega)(r)$. Then the derivative $f'(r) = df(r)/dr$ can be written as the derivative of the convolution, $f'(r) = (\tilde{g} * \omega')(r)$.*

Proof. Fix $r \in \mathbb{R}$. For $h \neq 0$, consider the quantity $\rho_h = \frac{1}{h}(f(r+h) - f(r))$, and note that $f'(r) = \lim_{h \rightarrow 0} \rho_h$. Recalling the definition of f as a convolution and using the mean-value theorem, we can write ρ_h as

$$\rho_h = \int_{-\infty}^{\infty} \tilde{g}(x) \left(\frac{\omega(r-x+h) - \omega(r-x)}{h} \right) dx = \int_{-\infty}^{\infty} g(x) \omega'(r-x+\xi_h) dx,$$

for some ξ_h between 0 and h . Let

$$\rho_0 := \int_{-\infty}^{\infty} \tilde{g}(x) \omega'(r-x) dx = (\tilde{g} * \omega')(r).$$

Then by another application of the mean-value theorem, we can write

$$\begin{aligned} \Delta_h &:= \rho_h - \rho_0 = \xi_h \int_{-\infty}^{\infty} \tilde{g}(x) \left(\frac{\omega'(r-x+\xi_h) - \omega'(r-x)}{\xi_h} \right) dx \\ &= \xi_h \int_{-\infty}^{\infty} \tilde{g}(x) \omega''(r-x+\xi_h^{(2)}) dx, \end{aligned} \tag{25}$$

for some $\xi_h^{(2)}$ lying between 0 and ξ_h . One can easily verify that the second derivative of ω is given by

$$\omega''(x) = \frac{q(x)}{c^2} \phi\left(\frac{x}{\sqrt{c}}\right),$$

where $q(x)$ is the polynomial $q(x) = (x^2 - c)p(x) - 2cxp'(x) + c^2p''(x)$. Since g is L -Lipschitz, for each $x \in \mathbb{R}$ we have

$$0 \leq \tilde{g}(x) = g(\exp(x)) \leq g(0) + |g(\exp(x)) - g(0)| \leq g(0) + L \exp(x)$$

This gives us the estimate

$$\begin{aligned} &\left| \int_{-\infty}^{\infty} \tilde{g}(x) \omega''(r-x+\xi_h^{(2)}) dx \right| \\ &\leq \frac{1}{c^2} \int_{-\infty}^{\infty} (g(0) + L \exp(x)) \cdot |q(r-x+\xi_h^{(2)})| \cdot \phi\left(\frac{r-x+\xi_h^{(2)}}{\sqrt{c}}\right) dx \\ &= \frac{1}{c^{3/2}} \int_{-\infty}^{\infty} (g(0) + L \exp(r+\xi_h^{(2)} - \sqrt{c}y)) \cdot |q(\sqrt{c}y)| \cdot \phi(y) dy < \infty, \end{aligned}$$

where in the computation above we have used the substitution $y = (r-x+\xi_h^{(2)})/\sqrt{c}$. The last expression above shows that the integral is finite, since we are integrating exponential and polynomial functions against the Gaussian density. Plugging this bound to (25) and recalling that $|\xi_h| \leq |h|$, we obtain

$$|\Delta_h| \leq |h| \cdot \left| \int_{-\infty}^{\infty} \tilde{g}(x) \omega''(r-x+\xi_h^{(2)}) dx \right| \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

Since $\Delta_h = \rho_h - \rho_0$, this implies our desired conclusion,

$$f'(r) = \lim_{h \rightarrow 0} \rho_h = \rho_0 = (\tilde{g} * \omega')(r).$$

□

We are now ready to prove Lemma 12.

Proof of Lemma 12. We proceed by induction on a . The base case $a = 0$ follows from the definition of U . Assume the statement holds for some $a \geq 0$; we prove it also holds for $a + 1$. Our strategy is to express U_{S^a} as a convolution, use Lemma 13 to differentiate the convolution, and write the result back as an expectation.

Fix $S, c > 0$ for the rest of this proof. Let $\tilde{g}(x) = g(\exp(x))$ and $\phi_c(x) = \phi(x/\sqrt{c})$. From the inductive hypothesis and the fact that $-Z \stackrel{d}{=} Z$, we have

$$\begin{aligned}
U_{S^a}(S, c) &= \frac{1}{S^a c^{a/2}} \mathbb{E} \left[p^{[a]}(-Z, \sqrt{c}) \cdot \tilde{g} \left(\log S - \frac{c}{2} - \sqrt{c}Z \right) \right] \\
&= \frac{1}{S^a c^{a/2}} \int_{-\infty}^{\infty} p^{[a]}(-x, \sqrt{c}) \cdot \tilde{g} \left(\log S - \frac{c}{2} - \sqrt{c}x \right) \cdot \phi(x) dx \\
&= \frac{1}{S^a c^{(a+1)/2}} \int_{-\infty}^{\infty} p^{[a]} \left(-\frac{y}{\sqrt{c}}, \sqrt{c} \right) \cdot \tilde{g} \left(\log S - \frac{c}{2} - y \right) \cdot \phi_c(y) dy \\
&= \frac{1}{S^a c^{(a+1)/2}} \int_{-\infty}^{\infty} \tilde{g} \left(\log S - \frac{c}{2} - y \right) \cdot \omega(y) dy \\
&= \frac{1}{S^a c^{(a+1)/2}} (\tilde{g} * \omega) \left(\log S - \frac{c}{2} \right),
\end{aligned}$$

where in the computation above we have used the substitution $y = \sqrt{c}x$, and we have defined the function

$$\omega(y) = p^{[a]} \left(-\frac{y}{\sqrt{c}}, \sqrt{c} \right) \cdot \phi \left(\frac{y}{\sqrt{c}} \right).$$

In particular, ω has derivative

$$\omega'(y) = -\frac{1}{\sqrt{c}} \left(p_x^{[a]} \left(-\frac{y}{\sqrt{c}}, \sqrt{c} \right) + \frac{y}{\sqrt{c}} p^{[a]} \left(-\frac{y}{\sqrt{c}}, \sqrt{c} \right) \right) \phi \left(\frac{y}{\sqrt{c}} \right).$$

Differentiating U_{S^a} with respect to S and using the result of Lemma 13 give us

$$\begin{aligned}
U_{S^{a+1}}(S, c) &= -\frac{a}{S^{a+1} c^{(a+1)/2}} (\tilde{g} * \omega) \left(\log S - \frac{c}{2} \right) + \frac{1}{S^{a+1} c^{(a+1)/2}} (\tilde{g} * \omega') \left(\log S - \frac{c}{2} \right) \\
&= \frac{1}{S^{a+1} c^{(a+1)/2}} \int_{-\infty}^{\infty} \tilde{g} \left(\log S - \frac{c}{2} - y \right) (\omega'(y) - a\omega(y)) dy \\
&= \frac{1}{S^{a+1} c^{a/2}} \int_{-\infty}^{\infty} \tilde{g} \left(\log S - \frac{c}{2} - \sqrt{c}x \right) (\omega'(\sqrt{c}x) - a\omega(\sqrt{c}x)) dx \\
&= \frac{1}{S^{a+1} c^{a/2}} \int_{-\infty}^{\infty} \tilde{g} \left(\log S - \frac{c}{2} - \sqrt{c}x \right) \frac{((-x-a\sqrt{c})p^{[a]}(-x, \sqrt{c}) - p_x^{[a]}(-x, \sqrt{c}))}{\sqrt{c}} \phi(x) dx \\
&= \frac{1}{S^{a+1} c^{(a+1)/2}} \int_{-\infty}^{\infty} \tilde{g} \left(\log S - \frac{c}{2} - \sqrt{c}x \right) p^{[a+1]}(-x, \sqrt{c}) \phi(x) dx \\
&= \frac{1}{S^{a+1} c^{(a+1)/2}} \mathbb{E} \left[p^{[a+1]}(-Z, \sqrt{c}) \cdot \tilde{g} \left(\log S - \frac{c}{2} - \sqrt{c}Z \right) \right] \\
&= \frac{1}{S^{a+1} c^{(a+1)/2}} \mathbb{E} \left[p^{[a+1]}(Z, \sqrt{c}) \cdot g \left(S \cdot \left(-\frac{c}{2} + \sqrt{c}Z \right) \right) \right],
\end{aligned}$$

as desired. In the computation above we have again used the substitution $x = y/\sqrt{c}$ and the fact that $-Z \stackrel{d}{=} Z$. This completes the induction step and the proof of the lemma. \square

As an example, the first few polynomials $p^{[a]}(x, y)$ are

$$\begin{aligned}
p^{[0]}(x, y) &= 1 \\
p^{[1]}(x, y) &= x \\
p^{[2]}(x, y) &= x^2 - yx - 1 \\
p^{[3]}(x, y) &= x^3 - 3yx^2 + (2y^2 - 3)x + 3y,
\end{aligned}$$

giving us the formulae

$$\begin{aligned}
U(S, c) &= \mathbb{E} \left[g \left(S \cdot \exp \left(-\frac{c}{2} + \sqrt{c}Z \right) \right) \right] \\
U_S(S, c) &= \frac{1}{S\sqrt{c}} \mathbb{E} \left[Z \cdot g \left(S \cdot \exp \left(-\frac{c}{2} + \sqrt{c}Z \right) \right) \right] \\
U_{S^2}(S, c) &= \frac{1}{S^2c} \mathbb{E} \left[(Z^2 - \sqrt{c}Z - 1) \cdot g \left(S \cdot \exp \left(-\frac{c}{2} + \sqrt{c}Z \right) \right) \right] \\
U_{S^3}(S, c) &= \frac{1}{S^3c^{3/2}} \mathbb{E} \left[(Z^3 - 3\sqrt{c}Z^2 + (2c - 3)Z + 3\sqrt{c}) \cdot g \left(S \cdot \exp \left(-\frac{c}{2} + \sqrt{c}Z \right) \right) \right].
\end{aligned}$$

We also have the following easy corollaries.

Corollary 2. For $a \geq 1$, $\mathbb{E}[p^{[a]}(Z, \sqrt{c})] = 0$. For $a \geq 2$, we also have $\mathbb{E}[p^{[a]}(Z + \sqrt{c}, \sqrt{c})] = 0$.

Proof. First assume $a \geq 1$, and take g to be the constant function $g(x) = 1$. In this case $U(S, c) = 1$ and $U_{S^a}(S, c) = 0$, so by the result of Lemma 12, $\mathbb{E}[p^{[a]}(Z, \sqrt{c})] = S^a c^{a/2} U_{S^a}(S, c) = 0$. Next, assume $a \geq 2$, and take g to be the linear function $g(x) = x$. In this case $U(S, c) = \mathbb{E}[S \cdot G(c)] = S$, so $U_{S^a}(S, c) = 0$. Then using the results of Lemma 11 and Lemma 12,

$$\mathbb{E} \left[p^{[a]}(Z + \sqrt{c}, \sqrt{c}) \right] = \exp \left(-\frac{c}{2} \right) \mathbb{E} \left[p^{[a]}(Z, \sqrt{c}) \exp(\sqrt{c}Z) \right] = S^{a-1} c^{a/2} U_{S^a}(S, c) = 0.$$

□

D.2 Calculations for the European-Option Payoff Function

In this section, we bound the derivatives $U_{S^a}(S, c)$ for the special case when g is the payoff function of the European call function, $g(x) = \max\{0, x - K\}$, where $K > 0$ is a constant. Note that the bounds on U_{S^3} and U_{S^4} are slightly stronger than the stated bounds (21) and (22), because in this case we are able to compute the derivatives exactly.

Lemma 14. Let $g(x) = \max\{0, x - K\}$. Then for all $S, c > 0$,

$$\begin{aligned}
|U_{S^2}(S, c)| &\leq \frac{K}{\sqrt{2\pi}} \cdot \frac{1}{S^2\sqrt{c}} \\
|U_{S^3}(S, c)| &\leq \frac{K}{\sqrt{2\pi}} \cdot \frac{(2\sqrt{c} + 1)}{S^3c} \\
|U_{S^4}(S, c)| &\leq \frac{K}{\sqrt{2\pi}} \cdot \frac{(6c + 5\sqrt{c} + 2)}{S^4c^{3/2}}
\end{aligned}$$

Proof. We first compute the Black-Scholes value $U(S, c)$. Define

$$\alpha \equiv \alpha(S, c) = -\frac{1}{\sqrt{c}} \log \frac{S}{K} + \frac{\sqrt{c}}{2},$$

and observe that $S \cdot \exp(-c/2 + \sqrt{c}Z) \geq K$ if and only if $Z \geq \alpha$. Then using the result of Lemma 11, we have

$$\begin{aligned}
U(S, c) &= \mathbb{E} \left[\left(S \cdot \exp \left(-\frac{c}{2} + \sqrt{c}Z \right) - K \right) \cdot \mathbf{1}\{Z \geq \alpha\} \right] \\
&= S \cdot \exp \left(-\frac{c}{2} \right) \mathbb{E} \left[\exp(\sqrt{c}Z) \cdot \mathbf{1}\{Z \geq \alpha\} \right] - K \mathbb{P}(Z \geq \alpha) \\
&= S \mathbb{P}(Z \geq \alpha - \sqrt{c}) - K \mathbb{P}(Z \geq \alpha) \\
&= S \Phi(-\alpha + \sqrt{c}) - K \Phi(-\alpha).
\end{aligned}$$

Differentiating the formula above with respect to c and applying the Black-Scholes differential equation (6), we get

$$U_{S^2}(S, c) = \frac{2}{S^2} U_c(S, c) = \frac{1}{S^2c} [S\alpha\phi(-\alpha + \sqrt{c}) + K(-\alpha + \sqrt{c})\phi(\alpha)] = \frac{K}{S^2\sqrt{c}} \phi(\alpha),$$

where the last equality follows from the relation $S\phi(-\alpha + \sqrt{c}) = K\phi(\alpha)$. In particular, we have the bound $0 \leq U_{S^2}(S, c) \leq K/(S^2\sqrt{2\pi c})$. A direct calculation reveals that the higher order derivatives of U are given by

$$U_{S^3}(S, c) = \frac{K}{S^3 c} (\alpha - 2\sqrt{c}) \phi(\alpha) \quad \text{and} \quad U_{S^4}(S, c) = \frac{K}{S^4 c^{3/2}} (\alpha^2 - 5\sqrt{c}\alpha + 6c - 1) \phi(\alpha).$$

It is not difficult to see that we have $|\alpha \exp(-\alpha^2/2)| \leq 1$ and $|\alpha^2 \exp(-\alpha^2/2)| \leq 1$. Applying these bounds to the formulae above gives us the desired conclusion. \square

D.3 Bounding the Second Derivative $U_{S^2}(S, c)$

We now bound the second-order derivative $U_{S^2}(S, c)$ in the general case.

Lemma 15. *Let $g: \mathbb{R}_0 \rightarrow \mathbb{R}_0$ be a convex, L -Lipschitz, K -linear function. Then for all $S, c > 0$,*

$$0 \leq U_{S^2}(S, c) \leq \frac{2LK}{\sqrt{2\pi}} \cdot \frac{1}{S^2\sqrt{c}}.$$

Proof. Recall that $U(S, c)$ is convex in S (Lemma 10), so $U_{S^2}(S, c) \geq 0$. If g is a linear function, say $g(x) = \gamma x$ for some $0 \leq \gamma \leq L$, then $U(S, c) = \mathbb{E}[\gamma S \cdot G(c)] = \gamma S$. In this case $U_{S^2}(S, c) = 0$, and we are done.

Now assume g is not a linear function. Since g is non-negative, L -Lipschitz, and K -linear, we can find $0 \leq \gamma \leq L$ such that $g'(x) = \gamma$ for $x \geq K$. Moreover, since g is convex and not a linear function, we also have that $\gamma > g'(0)$. Define the function $\tilde{g}: \mathbb{R}_0 \rightarrow \mathbb{R}_0$ by

$$\tilde{g}(x) = \frac{g(x) - g(0) - xg'(0)}{\gamma - g'(0)}, \quad (26)$$

and note that \tilde{g} is an increasing, 1-Lipschitz convex function with $\tilde{g}(0) = \tilde{g}'(0) = 0$, $0 \leq \tilde{g}'(x) \leq 1$, and $\tilde{g}'(x) = 1$ for $x \geq K$.

Consider the quantity $V(S, c) = \mathbb{E}[\tilde{g}(S \cdot G(c))]$, and note that we can write

$$V(S, c) = \frac{\mathbb{E}[g(S \cdot G(c)) - g(0) - g'(0) \cdot S \cdot G(c)]}{\gamma - g'(0)} = \frac{U(S, c) - g(0) - g'(0) \cdot S}{\gamma - g'(0)}.$$

Taking second derivative with respect to S on both sides and using the fact that $0 \leq \gamma - g'(0) \leq 2L$, we obtain

$$0 \leq U_{S^2}(S, c) = (\gamma - g'(0)) \cdot V_{S^2}(S, c) \leq 2L \cdot V_{S^2}(S, c).$$

We already know that $V_{S^2}(S, c) \geq 0$ since \tilde{g} is convex, so we only need to show that

$$V_{S^2}(S, c) \leq \frac{K}{\sqrt{2\pi}} \cdot \frac{1}{S^2\sqrt{c}}.$$

For $0 < S \leq K$, using the formula from Lemma 12 and the result of Lemma 16 below, we obtain

$$V_{S^2}(S, c) = \frac{1}{S^2 c} \mathbb{E} \left[(Z^2 - \sqrt{c}Z - 1) \cdot \tilde{g} \left(S \cdot \exp \left(-\frac{c}{2} + \sqrt{c}Z \right) \right) \right] \leq \frac{1}{S^2 c} \cdot \frac{S\sqrt{c}}{\sqrt{2\pi}} \leq \frac{K}{\sqrt{2\pi}} \cdot \frac{1}{S^2\sqrt{c}},$$

and for $S \geq K$, we use the result of Lemma 17 to obtain

$$V_{S^2}(S, c) = \frac{1}{S^2 c} \mathbb{E} \left[(Z^2 - \sqrt{c}Z - 1) \cdot \tilde{g} \left(S \cdot \exp \left(-\frac{c}{2} + \sqrt{c}Z \right) \right) \right] \leq \frac{1}{S^2 c} \cdot \frac{K\sqrt{c}}{\sqrt{2\pi}} = \frac{K}{\sqrt{2\pi}} \cdot \frac{1}{S^2\sqrt{c}}.$$

This completes the proof of the lemma. \square

It remains to prove the following two results, which we use in the proof of Lemma 15 above with \tilde{g} in place of g . Note that the first result below does not use the assumption that g is eventually linear.

Lemma 16. *Let $g: \mathbb{R}_0 \rightarrow \mathbb{R}_0$ be an increasing, nonnegative, convex, 1-Lipschitz function. Then for all $S, c > 0$,*

$$\mathbb{E} \left[(Z^2 - \sqrt{c}Z - 1) \cdot g \left(S \cdot \exp \left(-\frac{c}{2} + \sqrt{c}Z \right) \right) \right] \leq \frac{S\sqrt{c}}{\sqrt{2\pi}}.$$

Proof. Fix $S, c > 0$, and define the following quantities:

$$\begin{aligned} t_1 &= \frac{\sqrt{c} - \sqrt{c+4}}{2} & t_2 &= \frac{\sqrt{c} + \sqrt{c+4}}{2} \\ \lambda_1 &= S \cdot \exp\left(-\frac{c}{2} + \sqrt{c} t_1\right) & \lambda_2 &= S \cdot \exp\left(-\frac{c}{2} + \sqrt{c} t_2\right) \\ g_1 &= g(\lambda_1) & g_2 &= g(\lambda_2) \\ t_* &= \frac{1}{\sqrt{c}} \log\left(\exp(\sqrt{c} t_2) - \frac{1}{S} \cdot \exp\left(\frac{c}{2}\right) \cdot (g_2 - g_1)\right). \end{aligned}$$

Furthermore, define the function $h: \mathbb{R} \rightarrow \mathbb{R}_0$ by

$$h(x) = g_1 + \left(g_2 - g_1 - \lambda_2 + S \cdot \exp\left(-\frac{c}{2} + \sqrt{c} x\right)\right) \cdot \mathbf{1}\{x \geq t_*\}.$$

We will show that

$$\mathbb{E}\left[(Z^2 - \sqrt{c}Z - 1) \cdot g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}Z\right)\right)\right] \leq \mathbb{E}\left[(Z^2 - \sqrt{c}Z - 1) \cdot h(Z)\right], \quad (27)$$

and furthermore, we can evaluate the latter expectation explicitly:

$$\mathbb{E}\left[(Z^2 - \sqrt{c}Z - 1) \cdot h(Z)\right] = S\sqrt{c} \phi(t_* - \sqrt{c}) \leq \frac{S\sqrt{c}}{\sqrt{2\pi}}.$$

We begin by noting that t_1 and t_2 are the two roots of the polynomial $x^2 - \sqrt{c}x - 1$. Since g is increasing and 1-Lipschitz,

$$g_2 - g_1 = g(\lambda_2) - g(\lambda_1) \leq \lambda_2 - \lambda_1 = S \cdot \exp\left(-\frac{c}{2}\right) (\exp(\sqrt{c} t_2) - \exp(\sqrt{c} t_1)).$$

Therefore, from the definition of t_* , we see that

$$\exp(\sqrt{c} t_2) - \exp(\sqrt{c} t_*) = \frac{1}{S} \cdot \exp\left(\frac{c}{2}\right) \cdot (g_2 - g_1) \leq \exp(\sqrt{c} t_2) - \exp(\sqrt{c} t_1),$$

so $t_1 \leq t_* \leq t_2$. Furthermore, by construction,

$$S \cdot \exp\left(-\frac{c}{2} + \sqrt{c} t_*\right) = S \cdot \exp\left(-\frac{c}{2} + \sqrt{c} t_2\right) - (g_2 - g_1) = \lambda_2 - g_2 + g_1,$$

so $h(t_*) = g_1$. This means h is a continuous convex function of x (although we will not actually use this property). We will now show that pointwise,

$$\phi(x) \cdot (x^2 - \sqrt{c}x - 1) \cdot g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}x\right)\right) \leq \phi(x) \cdot (x^2 - \sqrt{c}x - 1) \cdot h(x). \quad (28)$$

We consider four cases:

- Suppose $x \leq t_1$, so $x^2 - \sqrt{c}x - 1 \geq 0$. Since g is increasing and nonnegative,

$$0 \leq g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}x\right)\right) \leq g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c} t_1\right)\right) = g_1 = h(x).$$

- Suppose $t_1 \leq x \leq t_*$, so $x^2 - \sqrt{c}x - 1 \leq 0$. Since g is increasing,

$$g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}x\right)\right) \geq g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c} t_1\right)\right) = g_1 = h(x) \geq 0.$$

- Suppose $t_* \leq x \leq t_2$, so $x^2 - \sqrt{c}x - 1 \leq 0$. Since g is increasing and 1-Lipschitz,

$$\begin{aligned} & g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}x\right)\right) \\ & \geq g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c} t_2\right)\right) + S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}x\right) - S \cdot \exp\left(-\frac{c}{2} + \sqrt{c} t_2\right) \\ & = g_2 - \lambda_2 + S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}x\right) = h(x) \geq 0. \end{aligned}$$

- Suppose $x \geq t_2$, so $x^2 - \sqrt{c}x - 1 \geq 0$. Since g is increasing and 1-Lipschitz,

$$\begin{aligned}
& g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}x\right)\right) \\
& \leq g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}t_2\right)\right) + S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}x\right) - S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}t_2\right) \\
& = g_2 - \lambda_2 + S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}x\right) = h(x).
\end{aligned}$$

Integrating (28) over $x \in \mathbb{R}$ gives us the desired inequality (27). Let us now evaluate the expectation on the right hand side of (27). A simple computation using the properties of $Z \sim \mathcal{N}(0, 1)$ gives us

$$\begin{aligned}
\mathbb{E}[(Z^2 - \sqrt{c}Z - 1) \cdot h(Z)] &= g_1 \mathbb{E}[(Z^2 - \sqrt{c}Z - 1)] \\
&\quad + (g_2 - g_1 - \lambda_2) \cdot \mathbb{E}[(Z^2 - \sqrt{c}Z - 1) \cdot \mathbf{1}\{Z \geq t_*\}] \\
&\quad + S \cdot \exp\left(-\frac{c}{2}\right) \cdot \mathbb{E}[(Z^2 - \sqrt{c}Z - 1) \exp(\sqrt{c}Z) \cdot \mathbf{1}\{Z \geq t_*\}] \\
&= (g_2 - g_1 - \lambda_2) \cdot (t_* - \sqrt{c}) \phi(t_*) + St_* \phi(t_* - \sqrt{c}) \\
&= -S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}t_*\right) \cdot (t_* - \sqrt{c}) \phi(t_*) + St_* \phi(t_* - \sqrt{c}) \\
&= -S(t_* - \sqrt{c}) \phi(t_* - \sqrt{c}) + St_* \phi(t_* - \sqrt{c}) \\
&= S\sqrt{c} \phi(t_* - \sqrt{c}),
\end{aligned}$$

as desired. \square

The following result is similar to Lemma 16, except that this result assumes the K -linearity of g and achieves a stronger result.

Lemma 17. *Let $g: \mathbb{R}_0 \rightarrow \mathbb{R}_0$ be an increasing, nonnegative, convex, 1-Lipschitz function with the property that $g'(x) = 1$ for $x \geq K$. Then for all $S \geq K$ and $c > 0$,*

$$\mathbb{E}\left[(Z^2 - \sqrt{c}Z - 1) \cdot g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}Z\right)\right)\right] \leq \frac{K\sqrt{c}}{\sqrt{2\pi}}.$$

Proof. This proof is similar in nature to the proof of Lemma 16, and we omit some of the details.

Case 1: Suppose $S \geq K \exp(\sqrt{c(c+4)}/2)$. Recall the European-option payoff function $g_{\text{EC}}(x) = \max\{0, x - K\}$ from Section 2, and note that the K -linearity of g implies $g(x) = g(K) + g_{\text{EC}}(x)$ for $x \geq K$. Using the fact that g is increasing and K -linear, we can show that for all $x \in \mathbb{R}$ we have

$$(x^2 - \sqrt{c}x - 1) \cdot g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}x\right)\right) \leq (x^2 - \sqrt{c}x - 1) \cdot \left\{g(K) + g_{\text{EC}}\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}x\right)\right)\right\}.$$

Integrating both sides above with $Z \sim \mathcal{N}(0, 1)$ in place of x and using the result of Lemma 14, we obtain

$$\begin{aligned}
& \mathbb{E}\left[(Z^2 - \sqrt{c}Z - 1) \cdot g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}Z\right)\right)\right] \\
& \leq \mathbb{E}\left[(Z^2 - \sqrt{c}Z - 1) \cdot \left\{g(K) + g_{\text{EC}}\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}Z\right)\right)\right\}\right] \\
& = \mathbb{E}\left[(Z^2 - \sqrt{c}Z - 1) \cdot g_{\text{EC}}\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}Z\right)\right)\right] \leq \frac{K\sqrt{c}}{\sqrt{2\pi}}.
\end{aligned}$$

Case 2: Suppose $K \leq S \leq K \exp(\sqrt{c(c+4)}/2)$. Define the following quantities:

$$t_0 = \frac{\sqrt{c}}{2} - \frac{1}{\sqrt{c}} \log \frac{S}{K - g(K) + g_1}, \quad \lambda_1 = S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}\left(\frac{\sqrt{c} - \sqrt{c+4}}{2}\right)\right),$$

and $g_1 = g(\lambda_1)$. Consider the function $h_2: \mathbb{R} \rightarrow \mathbb{R}_0$ given by

$$h_2(x) = g_1 + \left(g(K) - g_1 - K + S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}x\right)\right) \cdot \mathbf{1}\{x \geq t_0\}.$$

Using the fact that g is increasing, 1-Lipschitz, and K -linear, we can show that for all $x \in \mathbb{R}$,

$$(x^2 - \sqrt{c}x - 1) \cdot g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}x\right)\right) \leq (x^2 - \sqrt{c}x - 1) \cdot h_2(x).$$

Integrating both sides above with $Z \sim \mathcal{N}(0, 1)$ in place of x , we get

$$\mathbb{E}\left[(Z^2 - \sqrt{c}Z - 1) \cdot g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}Z\right)\right)\right] \leq \mathbb{E}\left[(Z^2 - \sqrt{c}Z - 1) \cdot h_2(Z)\right].$$

Following the same calculation as in the proof of Lemma 16, we can evaluate the latter expectation to be

$$\mathbb{E}\left[(Z^2 - \sqrt{c}Z - 1) \cdot h_2(Z)\right] = (K - g(K) + g_1) \sqrt{c} \phi(t_0) \leq \frac{K\sqrt{c}}{\sqrt{2\pi}},$$

where the last inequality follows from the relation $0 \leq g(K) - g_1 \leq K - \lambda_1$, since g is increasing and 1-Lipschitz. \square

D.4 Bounding the Higher-Order Derivatives

We now turn to bounding the higher-order derivatives $U_{S^3}(S, c)$ and $U_{S^4}(S, c)$. Our strategy is to approximate the eventually linear payoff function g by the European-option payoff g_{EC} and applying the bounds for g_{EC} developed in Lemma 14.

Lemma 18. *Let $g: \mathbb{R}_0 \rightarrow \mathbb{R}_0$ be a convex, L -Lipschitz, K -linear function. Then for all $S, c > 0$,*

$$\begin{aligned} |U_{S^3}(S, c)| &\leq 7LK \cdot \frac{\max\{c^{-3/2}, c^{-1/2}\}}{S^3}, \\ |U_{S^4}(S, c)| &\leq 28LK \cdot \frac{\max\{c^{-2}, c^{-1/2}\}}{S^4}. \end{aligned}$$

Proof. Since g is L -Lipschitz and K -linear, we can find $0 \leq \gamma \leq L$ such that $g(x) = g(K) + \gamma(x - K)$ for $x \geq K$. We decompose g into two parts,

$$g(x) = \gamma g_{\text{EC}}(x) + g^*(x),$$

where $g_{\text{EC}}(x) = \max\{0, x - K\}$ is the European-option payoff function, and $g^*: \mathbb{R}_0 \rightarrow \mathbb{R}_0$ is given by $g^*(x) = g(x)$ for $0 \leq x \leq K$, and $g^*(x) = g(K)$ otherwise.

Then the Black-Scholes value $U(S, c)$ also decomposes,

$$U(S, c) = \mathbb{E}[g(S \cdot G(c))] = \gamma \mathbb{E}[g_{\text{EC}}(S \cdot G(c))] + \mathbb{E}[g^*(S \cdot G(c))] \equiv \gamma U^{\text{EC}}(S, c) + U^*(S, c),$$

and similarly for the derivatives,

$$U_{S^a}(S, c) = \gamma U_{S^a}^{\text{EC}}(S, c) + U_{S^a}^*(S, c), \quad a \geq 0. \quad (29)$$

For the function g_{EC} , Lemma 14 tells us that for all $S, c > 0$,

$$|U_{S^3}^{\text{EC}}(S, c)| \leq \frac{3K}{\sqrt{2\pi}} \cdot \frac{\max\{c^{1/2}, 1\}}{S^3 c}, \quad |U_{S^4}^{\text{EC}}(S, c)| \leq \frac{13K}{\sqrt{2\pi}} \cdot \frac{\max\{c, 1\}}{S^4 c^{3/2}}. \quad (30)$$

Now for the second function g^* , we use Lemma 12 to write

$$U_{S^a}^*(S, c) = \frac{1}{S^a c^{a/2}} \mathbb{E}\left[p^{[a]}(Z, \sqrt{c}) \cdot g^*\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}Z\right)\right)\right]. \quad (31)$$

Since $\mathbb{E}[p^{[a]}(Z, \sqrt{c})] = 0$ for $a \geq 1$ (Corollary 2), we may assume that $g(0) = 0$, so $g^*(0) = 0$ as well. Since g is L -Lipschitz, this implies

$$\sup_{x \in \mathbb{R}} |g^*(x)| = \max_{0 \leq x \leq K} |g(x)| \leq \max_{0 \leq x \leq K} Lx = LK.$$

Therefore, by applying triangle inequality and Cauchy-Schwarz inequality to (31), we get for $a \geq 1$,

$$|U_{S^a}^*(S, c)| \leq \frac{1}{S^a c^{a/2}} \mathbb{E}\left[|p^{[a]}(Z, \sqrt{c})| \cdot LK\right] \leq \frac{LK}{S^a c^{a/2}} \mathbb{E}\left[\left(p^{[a]}(Z, \sqrt{c})\right)^2\right]^{1/2}. \quad (32)$$

For $a = 3, 4$, we use the recursion (24) to compute the polynomials $p^{[a]}(Z, \sqrt{c})$, and we evaluate the expectation $\mathbb{E}[(p^{[a]}(Z, \sqrt{c}))^2]$. Plugging in this expectation to (32) with $a = 3$ gives us

$$|U_{S^3}^*(S, c)| \leq \frac{LK}{S^3 c^{3/2}} \cdot (4c^2 + 18c + 6)^{1/2} \leq \sqrt{28} \cdot LK \cdot \frac{\max\{c, 1\}}{S^3 c^{3/2}}. \quad (33)$$

Therefore, by combining the bound above with the first inequality in (30) and using the decomposition (29), we get the first part of our lemma,

$$|U_{S^3}(S, c)| \leq \frac{3}{\sqrt{2\pi}} \cdot LK \cdot \frac{\max\{c^{1/2}, 1\}}{S^3 c} + \sqrt{28} \cdot LK \cdot \frac{\max\{c, 1\}}{S^3 c^{3/2}} \leq 7LK \cdot \frac{\max\{c, 1\}}{S^3 c^{3/2}}.$$

A similar computation with $a = 4$ yields the second part of the lemma,

$$|U_{S^4}(S, c)| \leq \frac{13}{\sqrt{2\pi}} \cdot LK \cdot \frac{\max\{c, 1\}}{S^4 c^{3/2}} + \sqrt{518} \cdot LK \cdot \frac{\max\{c^{3/2}, 1\}}{S^4 c^2} \leq 28LK \cdot \frac{\max\{c^{3/2}, 1\}}{S^3 c^2}.$$

□